Completeness principle and quantum field theory on nonglobally hyperbolic spacetimes

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We analyse in details the problems which one faces trying to quantize a scalar field on the spacelike cylinder being the simple example of a spacetime with closed timelike curves. Our analysis brings to light the fact that the usual set of positive and negative frequency solutions of the field equation turns out to be incomplete. The consequence of this fact is that the usual formulation of quantum field theory breaks down on such a spacetime.
We postulate the completeness principle and build on its basis the modified quantization procedure. As an example, the Hadamard function and $\langle\phi^2\rangle$ for the scalar field on the spacelike cylinder are calculated. It is shown that the "naive" method of images gives the same results of calculation.

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I. INTRODUCTION

During the last few years there has been considerable interest in "time machine" physics, i.e., in classical and quantum dynamics on nonglobally hyperbolic spacetimes which contain closed timelike curves. One of the problems, which the concept of a time machine entails, is that the usual formulation of quantum field theory breaks down on these spacetimes and instead the question (both mathematical and conceptual) arises: what it might mean to quantize a field on a nonglobally hyperbolic spacetime?

So far there is no full understanding of this problem. First attempts to clarify it were undertaken by Kay. He, in the framework of algebraic approach, proposed the concept of F-locality and postulated the F-locality condition [1] which has to play a role of "minimal necessary conditions" [2] for any quantum field theory. The basis, the F-locality condition is founded on, is a supposition that "the laws in the small should coincide with the usual laws for quantum field theory on globally hyperbolic spacetimes" [1]. Spacetimes for which there exists a field algebra satisfying F-locality was called by Kay as F-quantum compatible. The importance of F-locality condition is in that it turns out to be very restrictive. In particular, the general results was recently proved by Kay, Radzikowski and Wald [2] showing that no spacetime containing a compactly generated Cauchy horizon can be F-quantum compatible. As a consequence, this means that the F-locality principle is *inconsistent* with the existence of spacetimes containing closed timelike curves (see further discussion of this problem in Refs. [3]). One may interpret it as evidence that time machines are unphysical (see Ref. [2]) and must therefore be removed from physics in agreement with Hawking's chronology protection conjecture [4]. On the other hand, the F-locality principle itself could still be revised [5], or one could consider some alternative approaches for constructing quantum field theory on nonglobally hyperbolic spacetimes. For example, recently Li and Gott [6] used the "naive" method of images to calculate a vacuum polarization in the region of Misner space, containing closed timelike curves.

In this work we analyse in details the problems which one faces trying to apply the standard canonical quantization method for fields on spacetimes with closed timelike curves, and build the modified quantization procedure based on the completeness principle.

The units $c = \hbar = 1$ are used through the paper.

II. GLOBALLY AND NONGLOBALLY HYPERBOLIC SPACETIMES

First of all, let us briefly discuss the notions of the *global hyperbolic* and *nonglobal hyperbolic* spacetimes (one could find more details in [7]). Let (M, g) be a spacetime with a Lorentzian metric g. The metric determines the causal structure of spacetime by determining which curves are spacelike, which are null and which are timelike at each point.

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Locally, the metric will lead to a well defined light cone just as in Minkowski space. However, the global restrictions enforced by the assumption of an everywhere Lorentzian metric are rather weak and still permit a wide variety of more or less exotic global causal behaviour. The important distinction for our purposes is between the global hyperbolic and nonglobal hyperbolic spacetimes. We shall define a globally hyperbolic spacetime to be a time-orientable spacetime which possesses a Cauchy surface. Here, we define a Cauchy surface to be a smooth spacelike surface no two points of which can be joined by a causal curve (i.e. by a timelike or null curve), but such that every inextendible causal curve in the spacetime intersects it precisely once. Note that once there is one Cauchy surface in the spacetime, there will be many. In particular, there will exist many global choices of time coordinate (so a globally hyperbolic spacetime always arises as the product of the real line with some 3-manifold) whose constant time surfaces are all Cauchy.

A spacetime which does not possess a Cauchy surface will be called nonglobally hyperbolic. The simplest example of a nonglobally hyperbolic spacetime, which we shall examine in the next section, is the two dimensional spacelike cylinder.

III. QUANTIZED SCALAR FIELD ON THE SPACELIKE CYLINDER

Consider the Minkowski plane with the metric

$$ds^2 = -dt^2 + dx^2. \tag{1}$$

Choose a strip $\{t \in (-\infty, \infty), x \in [0, a]\}$ on this plane and assume that points on the bounds γ^-: $x = 0$ and γ^+: $x = a$ are to be identified, so that $(t, 0) \equiv (t, a)$. After this procedure we obtain a spacetime with the topology $S^1 \times R^1$, called the timelike cylinder $\mathcal{T}_2$. Note that $\mathcal{T}_2$ possesses a Cauchy surface; for example, a hypersurface $t = \text{const}$ is Cauchy. So the timelike cylinder is a globally hyperbolic spacetime.

Choose now a strip $\{t \in [0, a], x \in (-\infty, \infty)\}$ and 'glue' its bounds, so that $(0, x) \equiv (a, x)$. We obtain again a spacetime with the topology of cylinder: $S^1 \times R^1$, which shall be called the *spacelike cylinder* $\mathcal{S}_2$. It is obvious that the spacelike cylinder contains closed timelike curves; for example, the timelike curves $x = \text{const}$ are closed. So $\mathcal{S}_2$ is a nonglobally hyperbolic spacetime.

The spacetimes of cylinders $\mathcal{T}_2$ and $\mathcal{S}_2$ could be represented as factor spaces $\mathcal{T}_2 = M/\mathcal{R}_\mathcal{T}$ and $\mathcal{S}_2 = M/\mathcal{R}_\mathcal{S}$, respectively, where M is an universal covering space (the Minkowski plane), $\mathcal{R}_\mathcal{T}$ and $\mathcal{R}_\mathcal{S}$ are the equivalence relations

$$\begin{aligned} \mathcal{R}_\mathcal{T}: &\quad (t, x + a) \equiv (t, x), \\ \mathcal{R}_\mathcal{S}: &\quad (t + a, x) \equiv (t, x). \end{aligned} \tag{2}$$

Note that our two dimensional examples may be converted into four dimensional examples e.g. by taking the product with a flat two dimensional Euclidian space, so that $\mathcal{T}_4 = \mathcal{T}_2 \times R^2$ and $\mathcal{S}_4 = \mathcal{S}_2 \times R^2$.

Consider a scalar field on $\mathcal{S}_2$. Let ϕ be a real massless scalar field with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi. \tag{3}$$

The corresponding field equation in the metric (1) reads

$$[\partial_t^2 - \partial_x^2]\phi(t, x) = 0, \tag{4}$$

where $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$. In addition, it follows from the identification rule (2) and from the quadric form of the Lagrangian (3) that the scalar field has to obey the periodic condition

$$\phi(t + a, x) = \phi(t, x). \tag{5}$$

(For brevity we will not consider the antiperiodic condition $\phi(t + a, x) = -\phi(t, x)$, which is possible as well.) It is easy to see that the functions

$$e^{ik_n(x \pm t)}, \tag{6}$$

where

$$k_n = \frac{2\pi n}{a}, \quad n = \pm 1, \pm 2, \ldots,$$

form the set of particular solutions of the Eq. (4) obeying the condition (5).

Discuss now the problem of quantization of the scalar field on $\mathcal{S}_2$. In order to understand the essence of problem better, we shall first of all try to apply straightforwardly the standard scheme of the canonical quantization (we may refer, for example, to [8] for more details of the canonical quantization method). The main point of this approach is that a spacetime is manifestly divided into space and time. In the other words, the spacetime is foliated into spacelike hypersurfaces labelled by a constant value of the time coordinate t. Let Σ be a particular spacelike hypersurface $t = \text{const}$ with unit timelike normal vector $n^\mu = (\frac{\partial}{\partial t})^\mu = (1, 0)$. The canonical momentum π conjugated with ϕ is defined by

$$\pi = \frac{\delta \mathcal{L}}{\delta(\partial_t \phi)}.$$

The canonical quantization means that the field ϕ is regarded as an operator obeying the canonical commutation relations

$$\begin{aligned} &[\phi(t,x), \phi(t,x')] = 0, \\ &[\pi(t,x), \pi(t,x')] = 0, \\ &[\phi(t,x), \pi(t,x')] = i\delta(x,x'), \end{aligned} \tag{7}$$

where the commutators are calculated in a given spacelike hypersurface $t = \text{const}$ and $\delta(x, x')$ is a delta function with the property that

$$\int_{-\infty}^{\infty} \delta(x,x')dx = 1. \tag{8}$$

Now let us carry out the second quantization. With this aim we should construct an orthonormal set of positive and negative frequency modes being complete in a given spacelike hypersurface $t = \text{const}$, where a solution is defined to be positive frequency with respect to the timelike normal vector n^μ if $n^\mu \partial_\mu \phi = \partial_t \phi = -i\omega\phi$, $\omega > 0$. As usually, we divide the solutions (6) into positive and negative frequency modes ϕ_n and ϕ_n^*, respectively, where

$$\phi_n = C_n e^{-i\omega t + ik_n x}, \quad \omega = |k_n|. \tag{9}$$

The modes ϕ_n and ϕ_n^* are orthonormal provided $C_n = (4\pi\omega)^{-1/2}$, so that

$$\begin{aligned} &(\phi_n, \phi_{n'}) = \delta(k_n, k_{n'}), \\ &(\phi_n^*, \phi_{n'}^*) = -\delta(k_n, k_{n'}), \\ &(\phi_n^*, \phi_{n'}) = 0, \end{aligned} \tag{10}$$

where the scalar product of a pair of solutions f_1 and f_2 is defined as

$$(f_1, f_2) = -i \int_{-\infty}^{\infty} dx \big(f_1 \partial_t f_2^* - \partial_t f_1 \, f_2^*\big)_{t=\text{const}}. \tag{11}$$

Unfortunately, the set of modes $\{\phi_n, \phi_n^*\}$ is not possessing the important property; namely, it is *incomplete* in a given spacelike hypersurface $t = \text{const}$. This means that an arbitrary solution $\phi(t, x)$ cannot be represented as an expansion in terms of positive and negative frequency modes. To make sure of the incompleteness of the set $\{\phi_n, \phi_n^*\}$ it is enough to note that the necessary condition of completeness is not satisfied, i.e.,

$$\frac{1}{a} \sum_{n=-\infty}^{\infty} e^{ik_n(x-x')} = \sum_{n=-\infty}^{\infty} \delta(x - x' - na) \neq \delta(x,x'). \tag{12}$$

The incompleteness of the set of classical modes leads to the fact that the field operator ϕ cannot be in general represented as a sum $\sum_n (a_n \phi_n + a_n^\dagger \phi_n^*)$, where a_n and $a_n^\dagger$ are usual annihilation and creation operators. As a result, a natural definition of a vacuum state $|0\rangle$, such that $a_n|0\rangle = 0$, simply does *not* exist in this approach. Thus, we must conclude that the standard scheme of the canonical quantization turns out to be *inapplicable* and has to be modified.

The preceding analysis prompts to us at least two directions in which we might modify the standard quantization procedure. Discuss briefly first possibility.

Local vacuum

Thus, we have found that the problems of quantum field theory on nonglobally hyperbolic spacetimes are connected with the fact that the set $\{\phi_n, \phi_n^*\}$ of usual positive and negative frequency modes turns out to be incomplete in a given spacelike hypersurface Σ, and so there does not exist an expansion in terms of ϕ_n and ϕ_n^* for an arbitrary solution $\phi(t,x)$. Note that we mean a global expansion, i.e., that whose coefficients do not depend on a point of the hypersurface. At the same time, it is still possible to make such an expansion *locally*, i.e., at each separate point of Σ, so that $f(t,x)|_\Sigma = \sum_n [A_n(x)\phi_n + B_n(x)\phi_n^*]$, where the coefficients $A_n(x)$ and $B_n(x)$ are functions of a point of Σ. Let us stress that such the expansion has no universal character and can be realized in many ways with various coefficients $A_n(x)$ and $B_n(x)$. Nevertheless, assume that some choice of $A_n(x)$ and $B_n(x)$ has been done (for example, because of physical arguments). The corresponding field operator ϕ could be now represented as a sum $\sum_n [a_n(x)\phi_n + a_n^\dagger(x)\phi_n^*]$, where $a_n(x)$ and $a_n^\dagger(x)$ are the local annihilation and creation operators, respectively. A vacuum state $|0\rangle$, defined by $a_n(x)|0\rangle = 0$, $\forall n$, turns out to be depending on a point of the spacelike hypersurface Σ. In the other words, vacuum is in this case defined *locally*. This seems to be unsatisfactory with the physical point of view. Below we suggest another way to modify the quantization procedure which is free from this shortcoming.

The completeness principle

The completeness of a set of positive and negative frequency modes is an *essential* property of quantum field theory on a globally hyperbolic spacetime. We shall regard this property to be a fundamental feature of any quantum field theory and put it into the basis of the modified quantization procedure postulating

The completeness principle: *A spacetime has to be foliated into hypersurfaces (not necessarily spacelike) so that the corresponding set of positive and negative frequency modes would be complete.*

Let us apply this principle for the scalar field on the spacelike cylinder $\mathcal{S}_2$. To construct a complete orthonormal set of modes we choose a foliation of $\mathcal{S}_2$ into *timelike* hypersurfaces. Let Σ be a particular hypersurface $x = \text{const}$ with unit spacelike normal vector $n^\mu = (\frac{\partial}{\partial x})^\mu = (0,1)$. A canonical momentum π conjugated with ϕ should now be defined by

$$\pi = \frac{\delta \mathcal{L}}{\delta(\partial_x \phi)}. \tag{13}$$

As usual, we shall say that a solution is positive frequency with respect to the normal vector n^μ if $n^\mu \partial_\mu \phi = \partial_x \phi = -i\kappa\phi$, $\kappa > 0$. In our case the functions f_n and f_n^*, where

$$f_n = C_n e^{-i\kappa x + ik_n t}, \quad \kappa = |k_n|, \tag{14}$$

form the set $\{f_n, f_n^*\}$ of positive and negative frequency modes, respectively. One may check that the modes are orthonormal provided $C_n = (2a\kappa)^{-1/2}$, so that

$$(f_n, f_{n'}) = \delta_{nn'}, \quad (f_n^*, f_{n'}^*) = -\delta_{nn'}, \quad (f_n, f_{n'}^*) = 0, \tag{15}$$

where the scalar product is defined as

$$(f_1, f_2) = -i \int_0^a dt \big(f_1 \partial_x f_2^* - \partial_x f_1 \, f_2^*\big)_{x=\text{const}}. \tag{16}$$

(Note that the mode $n = 0$ should be excluded from the set $\{f_n, f_n^*\}$ because of non-normability.) The set $\{f_n, f_n^*\}$ is *complete* on a given timelike hypersurface $x = \text{const}$. To make sure of this we note that the necessary condition of completeness is satisfied, i.e.,

$$\frac{1}{a} \sum_{n=-\infty}^{\infty} e^{ik_n(t-t')} = \sum_{n=-\infty}^{\infty} \delta(t - t' - na) = \delta(t, t'), \tag{17}$$

where $\delta(t,t')$ is a delta function with the property that

$$\int_0^a \delta(t,t')dt = 1. \tag{18}$$

To quantize the scalar field we shall assume that ϕ is an operator obeying the "canonical" commutation relations

$$\begin{aligned} &[\phi(t,x), \phi(t',x)] = 0, \\ &[\pi(t,x), \pi(t',x)] = 0, \\ &[\phi(t,x), \pi(t',x)] = i\delta(t,t'), \end{aligned} \tag{19}$$

where the commutators are calculated on a given timelike hypersurface $x = \text{const}$. The field operator $\phi(t,x)$ may be represented as an expansion in terms of modes of the complete set $\{f_n, f_n^*\}$:

$$\phi(t,x) = \sum_n (b_n f_n + b_n^\dagger f_n^*) \tag{20}$$

Substituting this expansion into the Eqs. (19) gives the commutation relations for operators b_n and $b_n^\dagger$:

$$[b_n, b_{n'}] = 0, \quad [b_n^\dagger, b_{n'}^\dagger] = 0, \quad [b_n, b_{n'}^\dagger] = \delta_{nn'}. \tag{21}$$

The operators b_n and $b_n^\dagger$ act on orthonormal ket-vectors (Fock states) $|\ \rangle$ of Hilbert space $\mathcal{H}$. A vacuum state $|0\rangle$ is defined by $a_n|0\rangle = 0$, $\forall n$, and describes the situation when no particles are in a given timelike hypersurface $x = \text{const}$. The operator $b_n^\dagger$ determines a one-particle state $|1_n\rangle$ by $|1_n\rangle = b_n^\dagger|0\rangle$. Similarly, one may construct many-particle states (see e.g. Ref. [8]).

To make clear the physical sense of Fock states $|\ \rangle$ we consider the operator

$$H = \int_\Sigma T_{\mu\nu} n^\mu n^\nu d\Sigma, \tag{22}$$

where $T_{\mu\nu}$ is the stress-energy tensor of the scalar field, which could be obtained by the standard way in the following form:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha\phi\partial_\beta\phi. \tag{23}$$

Note that the expression (22) determines Hamilton or *total energy* operator for a globally hyperbolic spacetime with Σ being a Cauchy surface. Which sense get this operator in the considered case when $\mathcal{S}_2$ is a nonglobally hyperbolic spacetime and Σ is a timelike hypersurface? For a given hypersurface $x = \text{const}$ we obtain

$$H = \int_0^a T_{11} dt = \frac{1}{2} \int_0^a [(\partial_t\phi)^2 + (\partial_x\phi)^2] dt. \tag{24}$$

Substituting ϕ from (20) and taking into account the relations (21) we find

$$H = \sum_n (b_n^\dagger b_n + \frac{1}{2})\kappa, \tag{25}$$

The operator $N_n \equiv b_n^\dagger b_n$ is a particle number operator for mode n. Supposing that the energy of each quantum in mode n is equal to κ we find that H, given by Eq. (25), is a positive value total energy operator.

IV. HADAMARD FUNCTION AND $\langle\phi^2\rangle$

Consider now an example of calculation of a vacuum expectation value in a nonglobally hyperbolic spacetime. In this section we shall calculate the Hadamard function and the renormalized vacuum expectation value of field square $\langle\phi^2\rangle$ for the scalar field on the four dimensional timelike cylinder $\mathcal{S}_4$.

The spacetime $\mathcal{S}_4$ may be represented as the factor space $M/\mathcal{R}$, where M is Minkowski spacetime with the metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{26}$$

and $\mathcal{R}$ is the equivalence relation

$$(t+a,x,y,z) \equiv (t,x,y,z). \tag{27}$$

The scalar field ϕ with the Lagrangian (3) obeys the field equation

$$[\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2]\phi = 0 \tag{28}$$

and, additionally, has to satisfy the periodic condition

$$\phi(t+a,x,y,z) = \phi(t,x,y,z). \tag{29}$$

The particular solutions of the field equation (28) obeying the periodic condition (29) are

$$e^{\pm i\kappa x} e^{i(\omega_n t + k_y y + k_z z)}, \tag{30}$$

where

$$\omega_n = \frac{2\pi n}{a}, \quad \kappa = \sqrt{\omega_n^2 - k_y^2 - k_z^2}, \quad n = \pm 1, \pm 2, \ldots \tag{31}$$

Demanding κ to be real gives

$$k_y^2 + k_z^2 \leq \omega_n^2. \tag{32}$$

Choose a foliation of $\mathcal{S}_4$ into timelike hypersurfaces. Let Σ be a particular hypersurface $x = \text{const}$ with unit timelike normal vector $n^\mu = (\frac{\partial}{\partial x})^\mu = (0,1,0,0)$, which defines a positive frequency solution by $n^\mu \partial_\mu \phi = \partial_x \phi = -i\kappa\phi$, $\kappa > 0$. The positive frequency solutions may be taken in the following form:

$$f_J = \frac{1}{\sqrt{8\pi^2 a\kappa}} e^{-i\kappa x} e^{i(\omega_n t + k_y y + k_z z)}, \tag{33}$$

where $J \equiv \{n, k_y, k_z\}$. The functions f_J and f_J^* form the set $\{f_J, f_J^*\}$ of positive and negative frequency modes, which are orthonormal with respect to the scalar product

$$(f_1, f_2) = -i \int_0^a dt \iint dy\, dz (f_1 \partial_x f_2^* - \partial_x f_1\, f_2^*)_{x=\text{const}}. \tag{34}$$

The important property of the set $\{f_J, f_J^*\}$ is that it is complete on a given timelike hypersurface $x = \text{const}$.

The Hadamard function for the scalar field is defined as

$$G^{(1)}(X,X') = \langle 0|\phi(X)\phi(X') + \phi(X')\phi(X)|0\rangle, \tag{35}$$

where $X = (t, \mathbf{r}) = (t,x,y,z)$. Represent the field operator $\phi(X)$ as an expansion in terms of modes of the set $\{f_J, f_J^*\}$:

$$\phi(X) = \sum_J (b_J f_J + b_J^\dagger f_J), \tag{36}$$

where

$$\sum_J = \sum_{n=-\infty}^{\infty}{}' \iint_{k_y^2 + k_z^2 \leq \omega_n^2} dk_y\, dk_z \tag{37}$$

and the prime means that the mode $n = 0$ is excluded. (Note that the mode $n = 0$ should be excluded because of non-normability.) Substituting this expansion into the expression (35) and taking into account that $b_J|0\rangle = 0$, $\forall J$, we obtain

$$G^{(1)}(X,X') = \sum_J \left[f_J(X) f_J^*(X') + f_J^*(X) f_J(X') \right]. \tag{38}$$

By using the modes (33) the Hadamard function may be represented in the following form:

$$G^{(1)}_{\mathcal{S}}(X,X') = \frac{1}{4\pi^2 a}\sum_{n=-\infty}^{\infty}{}' \iint\limits_{k_y^2+k_z^2\le\omega_n^2} dk_y\, dk_z \frac{1}{\kappa}\cos(\kappa\triangle x - \omega_n\triangle t - k_y\triangle y - k_z\triangle z), \tag{39}$$

where $\triangle X = X - X'$ and the subscript $\mathcal{S}$ denotes the Hadamard function on $\mathcal{S}_4$. Carrying out the appropriate transformations of the above expression (see the appendix for more details) we finally obtain

$$G^{(1)}_{\mathcal{S}}(X,X') = -\frac{1}{4\pi a\triangle r}\left[\frac{\sin\dfrac{2\pi}{a}(\triangle t - \triangle r)}{1-\cos\dfrac{2\pi}{a}(\triangle t - \triangle r)} - \frac{\sin\dfrac{2\pi}{a}(\triangle t + \triangle r)}{1-\cos\dfrac{2\pi}{a}(\triangle t + \triangle r)}\right]. \tag{40}$$

As usual the renormalized Hadamard function is taken to be

$$G^{(1)\mathrm{ren}}_{\mathcal{S}}(X,X') = G^{(1)}_{\mathcal{S}}(X,X') - G^{(1)}_{M}(X,X'), \tag{41}$$

where $G^{(1)}_{M}(X,X')$ is the Hadamard function for the usual Minkowski vacuum [8]:

$$G^{(1)}_{M}(X,X') = -\frac{1}{2\pi^2}\frac{1}{(t-t')^2-(x-x')^2-(y-y')^2-(z-z')^2}. \tag{42}$$

The renormalized vacuum expectation value of the field square $\langle\phi^2\rangle_{\mathcal{S}}$, characterizing vacuum fluctuations on $\mathcal{S}_4$, is found as follows:

$$\langle\phi^2\rangle_{\mathcal{S}} = \lim_{X'\to X} G^{(1)\mathrm{ren}}_{\mathcal{S}}(X,X'). \tag{43}$$

Carring out simple calculations we obtain

$$\langle\phi^2\rangle_{\mathcal{S}} = -\frac{1}{6a^2}. \tag{44}$$

For comparison we write down the corresponding expression for $\langle\phi^2\rangle$ calculated for the scalar field on the 'usual' spacelike cylinder $\mathcal{T}_4$ [8]:

$$\langle\phi^2\rangle_{\mathcal{T}} = \frac{1}{6a^2}. \tag{45}$$

Note that the absolute values of $\langle\phi^2\rangle_{\mathcal{S}}$ and $\langle\phi^2\rangle_{\mathcal{T}}$ are equal, whereas their signs are opposite.

V. METHOD OF IMAGES

An example of a nonglobally hyperbolic spacetime is the Misner space (see e.g. [7]). The two-dimensional Misner space may be obtained by quotienting the region $t+x>0$ of Minkowski plane by a fixed Lorentz boost. As a manifold, it is again a cylinder, but the metric – while locally flat – differs globally from either the timelike or the spacelike cylinder. In fact, the cylinder is divided into two halves by a single closed null geodesic (the surface labelled $\tau=0$) and one can show that the open half cylinder for $\tau<0$ is conformally isometric to the timelike cylinder, while the open half cylinder for $\tau>0$ is conformally isometric to the spacelike cylinder. In the other words, the Misner space consists of a globally hyperbolic region ($\tau<0$) and a region of closed timelike curves ($\tau>0$) separated by the surface $\tau=0$, which is called the chronology horizon. A quantum field theory is well-defined in the first region, and we may refer to the pioneering work by Hiscock and Konkowski [9] and also to [10], where the vacuum polarization of a scalar field in the globally hyperbolic region of the Misner space has been calculated and discussed the behaviour of the renormalized stress-energy tensor near the chronology horizon. At the same time, since there is no well-defined quantum field theory in nonglobally hyperbolic spacetimes, so until recently there was no calculations of the vacuum polarization in the region of closed timelike curves of the Misner space. However, recently Li and Gott [6] represented such a calculation. They used *a priori* the method of images adapted to the case of spacetimes with closed timelike curves.

Apply the method of images for calculation of the vacuum polarization of the scalar field on $\mathcal{S}_4$. As was mentioned, the universal covering space of $\mathcal{S}_4$ is Minkowski space. The Hadamard function $G^{(1)}_{M}(X,X')$ for the usual Minkowski

vacuum is given by Eq. (42). Taking into account the periodicity in time (29) we may construct the Hadamard function on $\mathcal{S}_4$ as an image sum

$$G^{(1)}_{\mathcal{S}}(t,\mathbf{r};t',\mathbf{r}') = \sum_{n=-\infty}^{\infty} G^{(1)}_{M}(t,\mathbf{r};t'+na,\mathbf{r}')$$
$$= -\frac{1}{2\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(t-t'+na)^2-(x-x')^2-(y-y')^2-(z-z')^2}. \quad (46)$$

Using the formula (43), which gives the value of $\langle\phi^2\rangle$, we obtain after simple calculations

$$\langle\phi^2\rangle_{\mathcal{S}} = -\frac{1}{\pi^2 a^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{1}{6a^2}. \quad (47)$$

Note that we have obtained the result which coincides with Eq. (44). Thus, we may conclude that the method of images is consistent with the modified quantization procedure based on the completeness principle.

VI. CONCLUDING REMARKS

Summarizing, let us stress once more that the completeness principle plays an essential role in quantum field theory on nonglobally hyperbolic spacetimes. Mathematically, this simply supposes the existence of a complete orthonormal basis of solutions of classical field equations. In the paper we have revealed the fact that in order to construct such a basis in a spacetime with closed timelike curves one should consider a foliation of spacetime into timelike hypersurfaces. A given initial timelike hypersurface performs a role like that of a spacelike Cauchy surface in a globally hyperbolic spacetime. In particular, the scalar product of a pair of solutions is determined on the hypersurface, and the conjugate momentum of a field is defined by a spacelike unit vector normal to the hypersurface.

Note that, with the mathematical point of view, there is no principal difference between a field theory on globally or nonglobally hyperbolic spacetimes. One just supposes the initial hypersurface to be spacelike (a Cauchy surface) or timelike, respectively. However, the situation is quite different with the physical point of view. Really, as is known in the case of globally hyperbolic spacetimes one specifies field's initial values on the Cauchy surface at a given moment of time, and this does uniquely determine the following field's evolution in time. But in the case of a spacetime with closed timelike curves one should specify field's initial values on a timelike hypersurface. As a result, the field becomes to be determined at each moment of time, and hence there is *no* evolution in this case. Of course, this situation is a direct consequence of the existence of closed timelike curves.

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APPENDIX

Consider the expression (39) for $G^{(1)}_{\mathcal{S}}$ and carry out the following transformation:

$$\frac{1}{4\pi^2 a}\sum_{n=-\infty}^{\infty}{}' \iint\limits_{k_y^2+k_z^2\le\omega_n^2} dk_y\, dk_z \frac{1}{\kappa}\cos(\kappa\triangle x-\omega_n\triangle t-k_y\triangle y-k_z\triangle z)$$
$$= \frac{1}{2\pi^2 a}\sum_{n=1}^{\infty}\cos\omega_n\triangle t \iint\limits_{k_y^2+k_z^2\le\omega_n^2} dk_y\, dk_z \frac{1}{\kappa}\cos(\kappa\triangle x-k_y\triangle y-k_z\triangle z). \quad (A1)$$

To calculate the integral in the last expression we make a change of variables. Introduce the polar coordinates (k,φ) in the space of vectors $\mathbf{k}=(k_y,k_z)$, so that

$$k_y = k\cos\varphi,\ k_z = k\sin\varphi,$$
$$k = \sqrt{k_y^2 + k_z^2},\ dk_y dk_z = k\,dk\,d\varphi.$$

Assume also that the polar axis is directed along the vector $\mathbf{v} = (\triangle y, \triangle z)$, so that $\mathbf{kv} = kv\cos\varphi$, where $v = \sqrt{\triangle y^2 + \triangle z^2}$. Now we may transform the expression (A1) as follows:

$$\begin{aligned}
&\frac{1}{2\pi^2 a}\sum_{n=1}^{\infty}\cos\omega_n\triangle t\int_0^{\omega_n}\int_0^{2\pi}\frac{k\,dk\,d\varphi}{\sqrt{\omega^2-k^2}}\cos(\sqrt{\omega^2-k^2}\triangle x - kv\cos\varphi)\\
&= \frac{1}{\pi a}\sum_{n=1}^{\infty}\cos\omega_n\triangle t\int_0^{\omega_n}\frac{k\,dk}{\sqrt{\omega^2-k^2}}\cos\sqrt{\omega^2-k^2}\triangle x\,J_0(kv)\\
&= \frac{1}{\pi a\triangle r}\sum_{n=1}^{\infty}\cos\omega_n\triangle t\,\sin\omega_n\triangle r\\
&= -\frac{1}{2\pi a\triangle r}\sum_{n=1}^{\infty}\left[\sin\omega_n(\triangle t-\triangle r) - \sin\omega_n(\triangle t+\triangle r)\right],
\end{aligned} \tag{A2}$$

where $J_0(z)$ is the Bessel function and $\triangle r = \sqrt{\triangle x^2 + \triangle y^2 + \triangle z^2}$. To carry out the last transformations we used the following formulas (see e.g. [11, (2.5.40.10)] and [12, (2.12.21.6)]):

$$\int_{x_0}^{x_0+2\pi} e^{-iz\sin x}dx = 2\pi J_0(z),$$

$$\int_0^a x\frac{\cos b\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}J_0(cx)dx = \frac{\sin a\sqrt{b^2+c^2}}{\sqrt{b^2+c^2}}.$$

So, we obtain the next form of $G_{\mathcal{S}}^{(1)}$:

$$G_{\mathcal{S}}^{(1)}(X,X') = -\frac{1}{2\pi a\triangle r}\sum_{n=1}^{\infty}\big[\sin\omega_n(\triangle t-\triangle r) - \sin\omega_n(\triangle t+\triangle r)\big]. \tag{A3}$$

To calculate the series in (A3) we use the 'trick' with regularization

$$\sum_{n=1}^{\infty}\sin nx = \lim_{\lambda\to 0}\sum_{n=1}^{\infty}e^{-n\lambda}\sin nx$$

and apply the following formula [11]:

$$\sum_{n=1}^{\infty}e^{-n\lambda}\sin nx = \frac{1}{2}\frac{\sin x}{\cosh\lambda - \cos x}.$$

Finally, we find

$$G_{\mathcal{S}}^{(1)}(X,X') = -\frac{1}{4\pi a\triangle r}\left[\frac{\sin\frac{2\pi}{a}(\triangle t-\triangle r)}{1-\cos\frac{2\pi}{a}(\triangle t-\triangle r)} - \frac{\sin\frac{2\pi}{a}(\triangle t+\triangle r)}{1-\cos\frac{2\pi}{a}(\triangle t+\triangle r)}\right]. \tag{A4}$$

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